Image Restoration with Regularization Convex Optimization Approach

Abdolreza Rashno, Foroogh Sadat Tabataba, and Saeed Sadri

Abstract. In this paper, Tikhonov regularization with l-curve parameter estimation as convex optimization problem has been proposed in image restoration as a solution of ill-posed problem stem from sparse and large scale blurring matrix which has many singular values of different orders of magnitude close to the origin. Also, since the restored image is so sensitive to initial guess (start point) of optimization algorithm, a new schema for feasible set and feasible start point has been proposed. Some numerical results show the efficiency of the proposed algorithm in comparison with previous proposed methods.

Keywords: Image restoration; convex optimization; Tikhonov regularization; l-curve estimation.

1. Introduction

One of the fundamental tasks of Image processing is image restoration which could be expressed as reconstruction of images that are degraded and noised by moving imaging instrument, moving objects, environment, non-uniform illumination and additive noise. Mathematically, degrading process is formulated with point spread function (PSF) which is a function that specifies how pixels in the image are distorted. In this paper, matrix A is used as blurring matrix which is very large ill-determined rank and has many singular values of different orders of magnitude close to the origin. Because of this property of A, this kind of image restoration is a large-scale linear discrete ill-posed problem which could be solved by regularization methods. For this kind of problems some recently developed methods have been proposed such as Sylvester Tikhonov-regularization methods [1], convex constrained optimization for large-scale ill-conditioned generalized Sylvester equations [2], interior-point method for large constrained discrete ill-possed problems [3], enforcing non-negativity in image reconstruction algorithms [4] and interior-point trust-region-based method for large-scale non-negative regularization [5].

In this paper, the mentioned ill-posed problem is solved by regularization methods which have the desirable properties such as having the small norm and smoothness cost function. Regularization parameter is estimated by L-curve method [6].

Both regularization and blurring matrices are considered as a Kronecker product of two small matrices. Finally, since the convergence of such optimization problems is so sensitive to initial point of algorithm, the new schema is proposed for feasible start and feasible set which lead the algorithm to more accurate results.

The rest of this paper is organized as follows: Section 2 presents the recent proposed methods in image restoration, Section 3 describes some mathematical background of regularization convex optimization and l-curve parameter estimation. The new feasible start, feasible set and proposed algorithm for image restoration are all described in Section 4. Experimental setup and results are described in Section 5. Finally, the conclusion and future works are discussed in Section 6.

2. Related Works

There are many methods which have been proposed for image restoration task. In [1, 2, 7] Bouhamidi et al. proposed the image restoration technique as a large-scale linear discrete ill-posed problem with the right-hand side noise by regularization approach and image restoration method based on convex constrained optimization for large-scale ill-conditioned generalized Sylvester equations. In [8] Bredies et al. developed a total generalized variation (TGV) method which preserves the image edges with some smoothness in the regions away from edges. In [9] Zhue et al. proposed the effective restoration method for both smooth and non-smooth images based on mean curvature model. Their method has a drawback of its difficulty to being solved efficiently. In [10] Liu et al. extended the total variation with overlapping group sparsity for image restoration. They proposed a convex cost function and an efficient algorithm for solving the minimization problem. Also, in [11] Cai et al. proposed a method for image deblurring in tight frame domains which is reduced to finding a sparse solution of a system of linear equations with the rectangular coefficient matrix. Finally, in [12] Sun et al. proposed an augmented Lagrangian formulation with a special linearized fixed point iteration and a nonlinear multi-grid method for image restoration. One of the main challenges of iteration methods is the initial guess (primary point). In iteration methods, for different initial points, different results are achieved. Also, the inappropriate initial points (infeasible points) mislead the iteration algorithms. In this work, we address this challenge by the proposed algorithm for finding the feasible start (initial) point.

3. Regularization Convex Optimization Methods

Generally, image restoration problem can be formulated as the system of linear equations in equation (1),

$$g = Ax + n \tag{1}$$

Manuscript received May 27, 2014; revised August 20, 2014; accepted August 29, 2014.
The authors are with the Department of Electrical and Computer Engineering, Isfahan University of Technology, Isfahan, Iran. The corresponding author's email is: a.rashno@ec.iut.ac.ir

where x is a vector of size $(mn) \times 1$ which is achieved by stacking the rows of true image X of size $m \times n$ in equation (2).

$$x = (X_{11}, X_{12}, \dots, X_{1n}, X_{21}, \dots, X_{2n}, \dots, X_{m1}, \dots, X_{mn}) \quad (2)$$

Also, n is a vector of size $(mn) \times 1$ by stacking the rows of additive noise matrix N of size $(m \times n)$ and A is blurring matrix which is $(mn \times mn)$. Finally, g is a vector with $m \times n$ element achieved from degraded and noised available output image G of size $(m \times n)$. For any x, the lower differences between Ax and g, the better approximation could be achieved. So the problem of image restoration is formulated as a convex optimization problem

$$\min_{x \in FS} \|Ax - g\|, \quad (3)$$

where FS is a convex set and denoted as feasible set of the problem and$FS \in R^{mn}$. Suppose that we have the image of size 256×256, as discussed before the size of blurring matrix A is $(256 \times 256) \times (256 \times 256) = 65536 \times 65536$ which is extremely large and the direct solution of (3) involves high order computation task. For such problems, the Kronecker product takes a direct effect in this issue by sparse factorization of the blurring matrix. Let H be a $(m \times n)$ matrix and K be a $(p \times q)$ matrix. The Kronecker product of H and K is $H \otimes K = (h_{ij}K)$ and $h_{ij} \in H$ for all i and j. It means that each element of His multiplied to whole matrix K, so, the result of this Kronecker product is a $(mp) \times (nq)$ matrix.

As discussed, regularization methods play an important role in ill-posed problems with smoothing the cost function by adding an extra term weighted by γ regularization parameter. So the image restoration problem in (3) is converted to (4) which is regularization convex optimization problem,

$$\min_{x \in FS} (\|Ax - g\| + \gamma \|Rx\|), \quad (4)$$

where R is a regularization matrix. Both A and R, are considered as Kronecker product of two smaller matrix defined by$A = A_1 \otimes A_2$ and $R = R_1 \otimes R_2$. For better gradient estimation, (4) can be formulated as (5),

$$\min_{x \in FS} (\|Ax - g\|_2^2 + \gamma^2 \|Rx\|_2^2) \quad (5)$$

By applying the Kronecker product to A and R, (6) is achieved,

$$\min_{x \in FS} (\|(A_1 \otimes A_2)x - g\|_2^2 + \gamma^2 \|R_1 \otimes R_2 x\|_2^2) \quad (6)$$

Finally, considering the property for Kronecker product in equation (7),

$$\text{Vector}(AXB) = (B^T \otimes A)\text{Vector}(X) \quad (7)$$

The regularization optimization problem leads to (8),

$$\min_{x \in FS} (\|(A_1 X A_2{}^T - g\|_2^2 + \gamma^2 \|R_1 X R_2{}^T\|_2^2) \quad (8)$$

One of the most important issues in regularization method is the estimation of parameterγ. In this paper we apply L-curve method for this estimation. The L-curve tries to get the regularization parameter by the analysis of the norm of the regularized solution $\|x_k\|$ and the corresponding residual norm $\|b - Ax_k\|$. The L-curve is a plot of $\varphi(\|x_k\|)$ versus $\varphi(\|b - Ax_k\|)$ which φ could be $\varphi(t) = t$, $\varphi(t) = \sqrt{t}$ and $\varphi(t) = \log(t)$. This curve is shaped like the letter "L" and the optimal point is the point with the maximum curvature which lies in the corner of letter "L" [6].

4. Proposed Algorithm

In this paper, the feasible set of optimization algorithm is considered as a sphere or box in equation (9) which is convex and leads (5) to be a convex optimization problem,

$$FS = \{X \in R^{m \times n} : L \le X \le U\} \quad (9)$$

where X is a$m \times n$ image matrix and L and U are lower and upper bound matrixes, respectively. The notation $\le$ between two matrix means that all elements in matrix L such as L_{ij} are smaller than corresponding elements in matrix X denoting by X_{ij}. This relation is vice versa for X and U. As it is clear that the optimization algorithm searches among whole feasible set to find the best image matrix X, the more probability for existence of relevant X in feasible set, the higher chance for finding the optimized X exists.

Both the degree of blurriness of the image and the amount of additive noise affects selection of feasible set. Our idea is that the interval of the feasible set is selected based on the fact that how images are deviated from their origin. The proposed feasible set is an interval around available blurred and noised image G. Firstly a 5×5 window around each pixel in G is selected. Then, mean and variance for each window is calculated with equations (10) and (11),

$$\overline{G}(i,j) = \frac{1}{25} \sum_{d=i-2}^{i+2} \sum_{s=j-2}^{j+2} G(d,s) \quad (10)$$

$$\text{Var}(i,j) = \frac{1}{25} \sum_{d=i-2}^{i+2} \sum_{s=j-2}^{j+2} [G(d,s) - \overline{G}(d,s)]^2 \quad (11)$$

The lower and upper bounds for feasible set are proposed in equations (12) and (13),

$$L(i,j) = \begin{cases} G(i,j) - \propto . \text{Var}(i,j) & \text{if } \left(G(i,j) - \propto \text{Var}(i,j)\right) < 0 \\ 0 & \text{otherwise} \end{cases} \quad (12)$$

$$U(i,j) = G(i,j) + \propto \text{Var}(i,j) \quad (13)$$

The higher amount of noise added to image, the bigger interval of feasible set must be selected. The main issue for this idea is that the model of noise is unavailable. We address this problem by constant parameter α which controls the interval of feasible set. The feasible set interval grows linearly as the α parameter leads to bigger amounts. In our experiment, the best selection for feasible start point is G which is in the middle part of the feasible set, so this point is strictly feasible. The final algorithm for image restoration is presented as following:

Algorithm 1. Tikhonov regularization feasible start for image restoration

1. Initialize the X_0 (available blurred and noised image G), tolerance and maximum iteration.
2. Estimate the regularization parameter γ with L-curve method described in Section 2.
3. Repeat :
4. If $\nabla f(X_{ij}) \geq 0$, $\overline{X}_{ij} = L_{ij} else \overline{X}_{ij} = U_{ij}$
5. $\eta = < \nabla f(X)|\overline{X} - X >_F$
6. Find the ω^* which minimize the $\min_{\omega}(f(X + \omega(\overline{X} - X))$
7. Update $X = X + \omega^*(\overline{X} - X)$
8. Until max iteration is achieved or $\eta < Tolerance$

$f(X)$ is objective function of equation (5). $< A|B >_F$ is a type of inner product which is equal to $trace(A^T B)$. It could be implied that the well known Frobenius norm $\|.\|_F$ is given by $\|A\|_F = \sqrt{< A|A >_F}$. Also, the property in equation (14) could be implied.

$$< A|B >_F = < Vector(A)|Vector(B) >_2 \quad (14)$$

Step 4 is a solution of the $\min_X < \nabla f(X)|X >_F$ and $\nabla f(X)$ is computed from the equation (15) [7],

$$\nabla f(X) = 2A_1^T(A_1 X A_2^T - G)A_2 + 2\gamma^2 R_1^T R_1 X R_2^T R_2 \quad (15)$$

The solution of linear minimization problem in step 6 is stem from Tikhonov regularization solution in equation (16) [7],

$$\omega^* = \begin{cases} -\frac{<\nabla f(X)|\overline{X}-X>_F}{2\|A_1(\overline{X}-X)A_2^T\|_F^2 + 2\gamma^2\|R_1(\overline{X}-X)R_2^T\|_F^2} \\ \quad if \quad (-\frac{<\nabla f(X)|\overline{X}-X>_F}{2\|A_1(\overline{X}-X)A_2^T\|_F^2 + 2\gamma^2\|R_1(\overline{X}-X)R_2^T\|_F^2}) \leq 1 \\ 1 \quad otherwise \end{cases} \quad (16)$$

5. Experimental Setup and Results

To illustrate the effectiveness of the proposed algorithm, some numerical tests are performed. The proposed algorithm is implemented on a machine with 2.26 GHz Corei7 CPU and 6 GB of RAM and Windows 7 with two images named as fruit and cameraman with the sizes of 512x512 and 256x256 pixels respectively. The results are compared with the proposed algorithm in [7] and reduced Newton (RN) algorithm [13].First, the image is blurred by blur matrix A consisting of Kroncker product of A_1 and A_2. These two matrixes are considered to be equal and are computed with equation (17),

$$A_1(i,j) = A_2(i,j) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(i-j)^2}{2\sigma^2}\right) & if \quad |i-j| \leq r \\ 0 & otherwise \end{cases} \quad (17)$$

Then, the additive noise is added to blur noise-free image with white Gaussian noise of mean 0 and different variances. The regularization matrix R is computed from Kroncker product of R_1 and R_2. As in papers used [1, 3, 7], R_1 is tridiagonal matrix of [-1 2 -1] and R_2 is identity matrix. Fig. 1 and Fig. 2 show the results of both proposed and RN methods in cameraman and fruit images with two types of σ, r and noise variances.

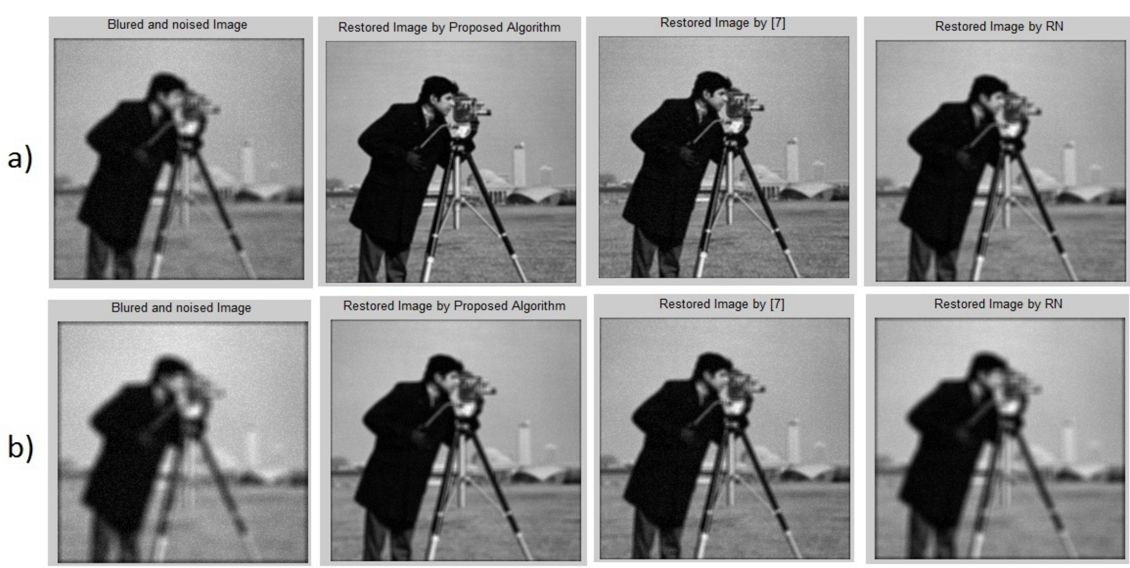


Fig. 1. Restored cameraman images by proposed method and reduced Newton: a) Noise Variance = 2, Sigma = 4.4, r = 5 and b) Noise Variance = 2.5, Sigma = 8.4, r = 7.

There are some measures for evaluating the restored image result. In this research we have used Improvement Signal to Noise Ratio (ISNR), Mean Absolute Error (MAE), Mean Squared Error (MSE) and relative error measures which are computed by equations (18), (19), (20), and (21), respectively:

$$\mathrm{ISNR} = 10\log_{10}\left(\frac{\sum_{i,j}[g(i,j)-f(i,j)]^2}{\sum_{i,j}[\hat{f}(i,j)-f(i,j)]^2}\right) \quad (18)$$

$$\mathrm{MAE} = \frac{\sum_{i,j}|\hat{f}(i,j)-f(i,j)|}{MN} \quad (19)$$

$$\mathrm{MSE} = \frac{\sum_{i,j}|\hat{f}(i,j)-f(i,j)|^2}{MN} \quad (20)$$

$$\text{Relative Error}(X) = \frac{\|\hat{f}-f\|_F}{\|\hat{f}\|_F} \quad (21)$$

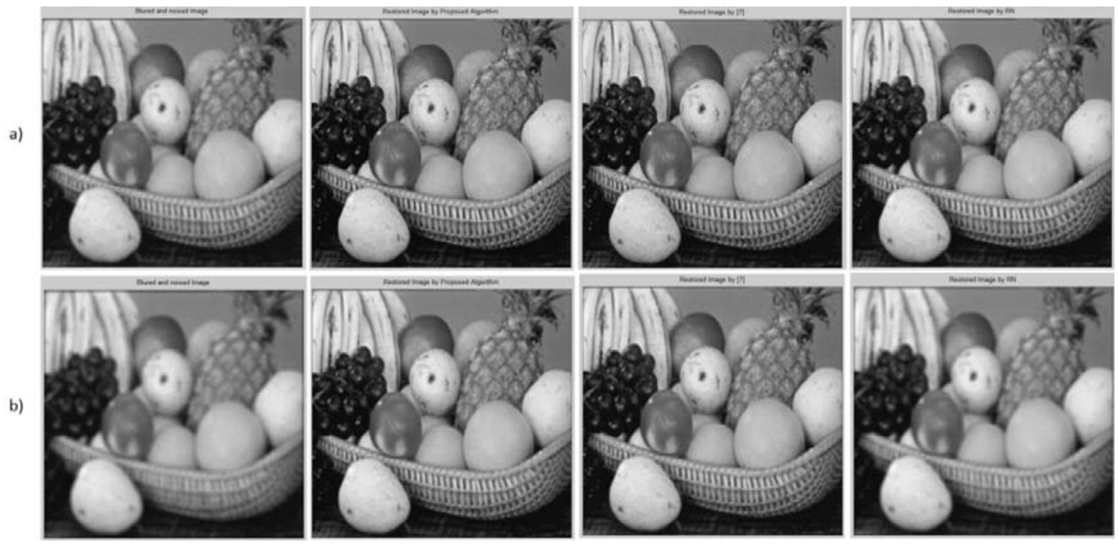


Fig. 2. Restored fruit images by proposed method and reduced Newton: a) Noise Variance = 2, Sigma = 4.4, r = 5 and b) Noise Variance = 2.5, Sigma = 8.4, r = 7.

Table 1. Results for Our Proposed Method, Proposed Method in [7] and RN.

	σ	r	Noise Variance	α	γ	Method	Relative Error	ISNR	MAE	MSE
Cameraman	1.4	3	1	15	0.954	Proposed Algorithm	0.1831	0.0514	7.3421	121.4
	1.4	3	1	-	-	Proposed Method in [7]	0.1812	0.0541	7.0134	118.3
	1.4	3	1	-	-	RN	0.2065	0.0432	7.9876	125.7
	4.4	5	2	15	0.954	Proposed Algorithm	0.2418	0.0332	9.1245	135.5
	4.4	5	2	-	-	Proposed Method in [7]	0.2634	0.0304	10.2345	141.9
	4.4	5	2	-	-	RN	0.2712	0.0301	10.8733	149.3
	8.4	7	2.5	15	0.954	Proposed Algorithm	0.3243	0.0118	13.4564	161.5
	8.4	7	2.5	-	-	Proposed Method in [7]	0.3294	0.0113	13.8734	168.6
	8.4	7	2.5	-	-	RN	0.3283	0.0104	14.6534	174.5
	12.4	10	5	15	0.954	Proposed Algorithm	0.4154	-0.0245	17.7645	191.4
	12.4	10	5	-	-	Proposed Method in [7]	0.4834	-0.0735	21.4533	205.3
	12.4	10	5	-	-	RN	0.5521	-0.1355	28.8734	213.5
Fruit	1.4	3	1	15	0.954	Proposed Algorithm	0.2123	0.0412	8.4633	127.3
	1.4	3	1	-	-	Proposed Method in [7]	0.2059	0.0443	7.9853	125.6
	1.4	3	1	-	-	RN	0.2354	0.0382	8.8753	135.1
	4.4	5	2	15	0.954	Proposed Algorithm	0.2534	0.0402	10.1342	139.7
	4.4	5	2	-	-	Proposed Method in [7]	0.2848	0.0356	11.1546	146.4
	4.4	5	2	-	-	RN	0.2914	0.0312	11.8912	153.2
	8.4	7	2.5	15	0.954	Proposed Algorithm	0.3341	0.0184	13.7564	165.5
	8.4	7	2.5	-	-	Proposed Method in [7]	0.3387	0.0112	14.1654	172.1
	8.4	7	2.5	-	-	RN	0.3467	0.0091	14.8745	179.7
	12.4	10	5	15	0.954	Proposed Algorithm	0.3918	-0.0341	16.3452	200.3
	12.4	10	5	-	-	Proposed Method in [7]	0.4453	-0.0674	18.8435	209.9
	12.4	10	5	-	-	RN	0.4976	-0.1613	25.5432	221.2

where f, $\hat{f}$ and g are original image, restored image and degraded image, respectively. Also, M and N are the image dimensions. It is worth mentioning that the higher amount of ISNR, the better restoration result is achieved while for MAE, MSE and relative error this relation is vice versa.

The γ regularization parameter, computed by L-curve, is 0.954 in its best case at the corner of curve, so we set this parameter to 0.954 as a constant for all experiments. Also, the best case for feasible set parameter $\propto$ was 15 which is a trade-off between time order of the algorithm and appropriate restoration results. Experiments are performed for different parameters of the algorithm and then results are shown in the table 1. As it is indicated, 4 types of noise and blur parameters include 1. $\sigma = 1.4$, r = 3 and Noise Variance = 1, 2. $\sigma = 4.4$, r = 5 and Noise Variance = 2, 3. $\sigma = 8.4$, r = 7 and Noise Variance = 2.5 and 4. $\sigma = 12.4$, r = 10 and Noise Variance = 5 are applied to cameraman and fruit images. As it is clear from the table 1, our proposed algorithm outperforms RN in all noise and blur cases and outperforms [7] in 2, 3 and 4 cases.

6. Conclusion

In this paper, a new schema for image restoration based on Tikhonov regularization convex optimization method has been proposed. The regularization parameter was estimated by L-curve method. The iteration methods in image restoration are so sensitive to initial point. The different initial points lead the algorithms to different results. Also, inappropriate initial points mislead the algorithms. In our proposed algorithm, the appropriate feasible set and feasible start point for the mentioned optimization problem was introduced which clearly affects both convergence and founded optimal solution of the algorithm. Our idea for feasible start point is very simple as well as it is strictly feasible since it is located in the middle of the feasible set. The results showed that the performance of new algorithm is relatively better in comparison with previous methods in the term of ISNR, MAE, MSE and relative error measures. For future trend, more advanced methods, such as generalized cross-validation methods could be tested on the new algorithm. Also, the more sophisticated non-convex feasible sets may lead to better results. Finally, the particle swarm optimization (PSO) method could be adapted for estimation of optimalγ parameter.

References

[1] A. Bouhamidi, and K. Jbilou, "Sylvester Tikhonov-regularization methods in image restoration,"*Journal of Computational and Applied Mathematics,* vol. 206, no. 1, pp. 86-98, 2007.

[2] A. Bouhamidi, K. Jbilou, and M. Raydan, "Convex constrained optimization for large-scale ill-conditioned generalized Sylvester equations," *Computational Optimization and Applications,* vol. 48, No. 2, pp. 233-253, 2011.

[3] S. Morigi, L. Reichel, and F. Sgallari, "An interior-point method for large constrained discrete ill-possed problems,"*Journal of Computational and Applied Mathematics,* vol. 233, pp. 1288-1297, 20010.

[4] J. Nagy and Z. Strakos, "Enforcing nonnegativity in image reconstruction algorithms," in: David C. Wilson, et al. (Eds.), Mathematical Modeling, Estimation, and Imaging, *Proceedings of SPIE,* vol. 4121, pp. 182-190, 2000.

[5] M. Rojas, and T. Steihaug, "An interior-point trust-region-based method for large-scale non-negative regularization," *Inverse Problems,* vol. 18, pp. 1291-1307, 2002.

[6] D. Calvetti, S. Morigi, L. Reichel, and F. Sgallari, "Tikhonov regularization and the L-curve for large discrete ill-posed problems," *Journal of Computational and Applied Mathematics,* vol. 123, pp. 423-446, 2000.

[7] A. Bouhamidi, R. Enkhbat, and K. Jbilou, "Conditional gradient Tikhonov method for a convex optimization problem in image restoration," *Journal of Computational and Applied Mathematics,* vol. 255, pp. 580-592, 2014.

[8] K. Bredies, K. Kunisch, and T. Pock, "Total generalized variation," *SIAM J. Imag. Sci,*vol. 3, pp. 492-526, 2010.

[9] W. Zhu and T.F. Chan, "Image denoising using mean curvature,"*SIAM J. Imag. Sci.,*vol. 5,pp. 1-32, 2012.

[10] J. Liu, T. Z. Huang,I. W. Selesnick, X. G. Lv, and P. Y. Chen, "Image restoration using total variation with overlapping group sparsity,"*Information Sciences,*vol. 295, pp. 232-246, 2015.

[11] J. F. Cai, S. Osher, and Z. W. Shen, "Linearized bregman iterations for frame-based image deblurring,"*SIAM J. Imag. Sci.,* vol. 2, pp. 226-252, 2009.

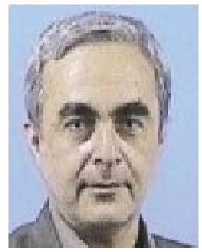
[12] L. Sun, and K. Chen, "A new iterative algorithm for mean curvature-based variational image denoising," *BIT Numerical Mathematics,* vol. 54, issue 2, pp. 523-553, 2013.

[13] B. Morini, M. Porcelli, and R. Chan, "A reduced Newton method for constrained linear least-squares problems," *Journal of Computational and Applied Mathematics,* vol. 233, pp. 2200-2212, 2010.

Abdolreza Rashno received the B.S. degree in computer engineering from Shahid Chamran University of Ahvaz, Iran, in 2009 and the M.S. degree (with honors) in artificial intelligence engineering from Kharazmi University of Tehran, Iran, in 2011.

Currently, he is a Ph.D. candidate of computer engineering at Isfahan University of Technology, Iran. His research interests are image segmentation, medical image analysis, image restoration, content-based image retrieval, evolutionary computing and machine learning.

Foroogh S. Tabataba received the B.S. degree (with honors) in electrical engineering from Isfahan University of Technology, Isfahan, Iran, in 2004 and the M.S. and the Ph.D. degrees in electrical engineering from Sharif University of Technology, Tehran, Iran, in 2006 and 2011, respectively.
From March 2004 to December 2011, she has been a Research Assistant at Advanced Communication Research Institute in Wireless Research Laboratory and Data Network Research Laboratory of the Electrical Engineering Department in Sharif University. In 2008, she has received a visiting fellowship from Australian National University and spent six months in ANU working with Dr. Sadeghi's group. In February 2012, she joined communication and networks group in the Department of Electrical and Computer Engineering at Isfahan University of Technology where she is now an assistant professor. She has won young professor grant of National Elite Foundation in September 2012. Her research interests include communication systems theory, wireless communications, cooperative networks, information theory applications, network coding and optical networks.



Saeed Sadri is an Associate Professor in Electrical and Computers Engineering Department, Isfahan University of Technology, Isfahan, Iran. He received the B.Sc. and the M.Sc. degrees in electrical engineering from the University of Tehran, Faculty of Engineering, Tehran, Iran, respectively in 1977 and 1979. He received the Ph.D. degree in communication engineering from the Isfahan University of Technology in 1997.